MULTIFRACTAL ANALYSIS: EXTREME EVENTS CUT-OFF, CORRELATIONS, LINEARIZATION EFFECT.

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Linearization effect

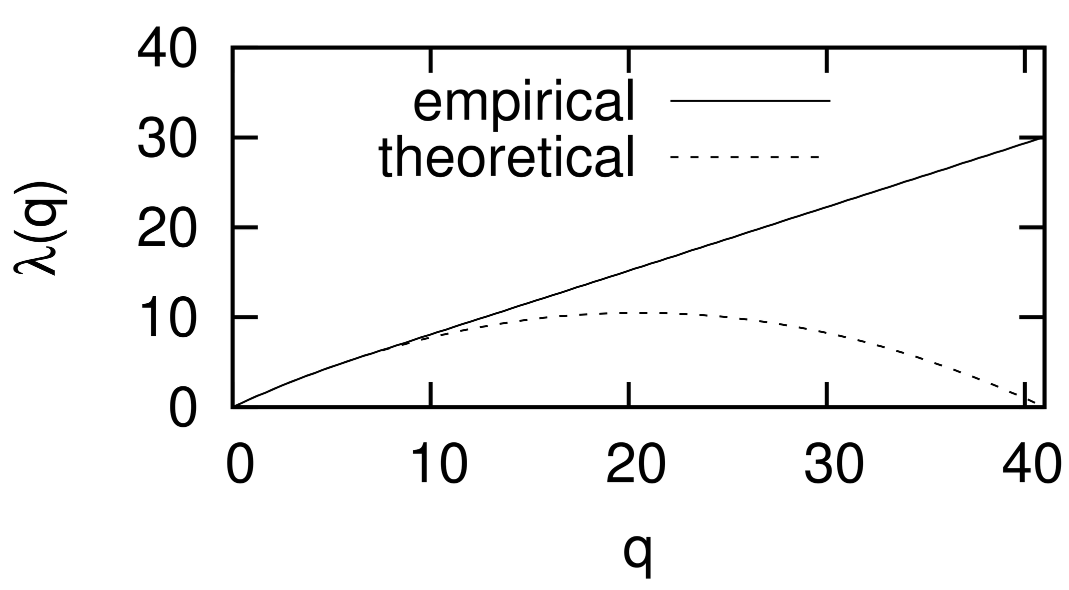
Multifractal Process $X(t)$ [?]:
Increments $T(a,t) = X(t+a) - X(t)$,

$$\langle T_a^q \rangle = a^{\lambda(q)},\ a \to 0^+$$

$\lambda(q)$ is related to the Legendre transform of the multifractal spectrum

Moments estimator: $S_n(a,q) = \frac{1}{n}\sum_{k=1}^{n} T(a,t_k)^q$

Linearization effect [?] Beyond a critical moment q^*, experimental measurement using S_n diverges from $\lambda(q)$.



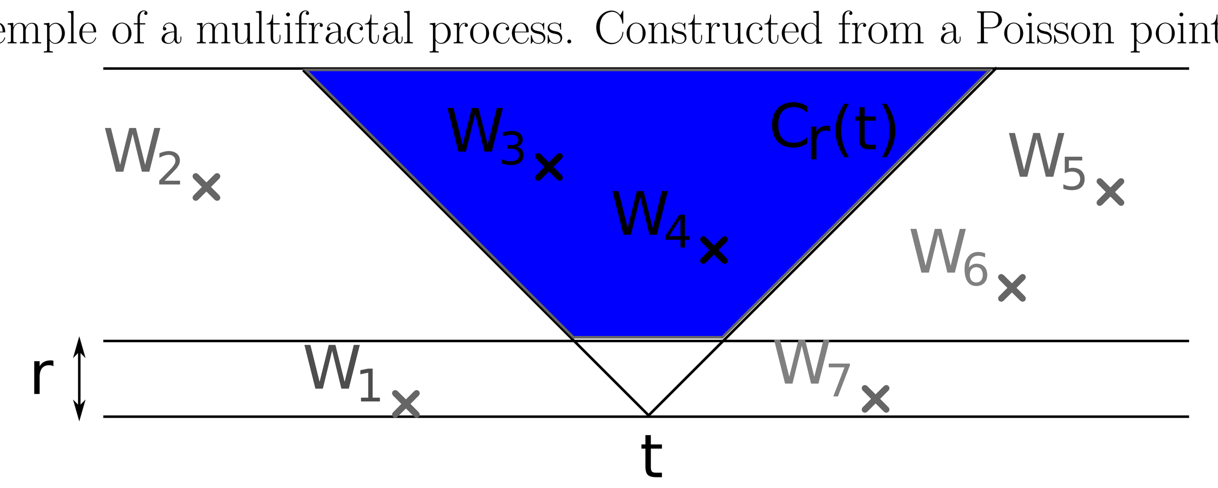
Large deviation properties
Large deviation theory [?] coupled with $\langle T_a^q \rangle = a^{\lambda(q)}$ indicates that the density probability p_a of $h_a = \frac{\ln T_a}{\ln a}$ verifies

$$p_a(h) \approx e^{-|\ln a|\psi(h)}$$

where $-\psi(h)$ is the Legendre transform of $\lambda(q)$.

Compound Poisson processes [?]

Standard exemple of a multifractal process. Constructed from a Poisson point process W_n



Compound Poisson Cascade

$$Q_r(t) = B_r(t) \prod_{(t_i,r_i)\in\mathcal{C}_r(t)} W_i,$$

where only multipliers associated with points belonging to the cone $\mathcal{C}_r(t)$ are taken into account and $B_r(t)$ is a normalising constant.

Compound Poisson Motion

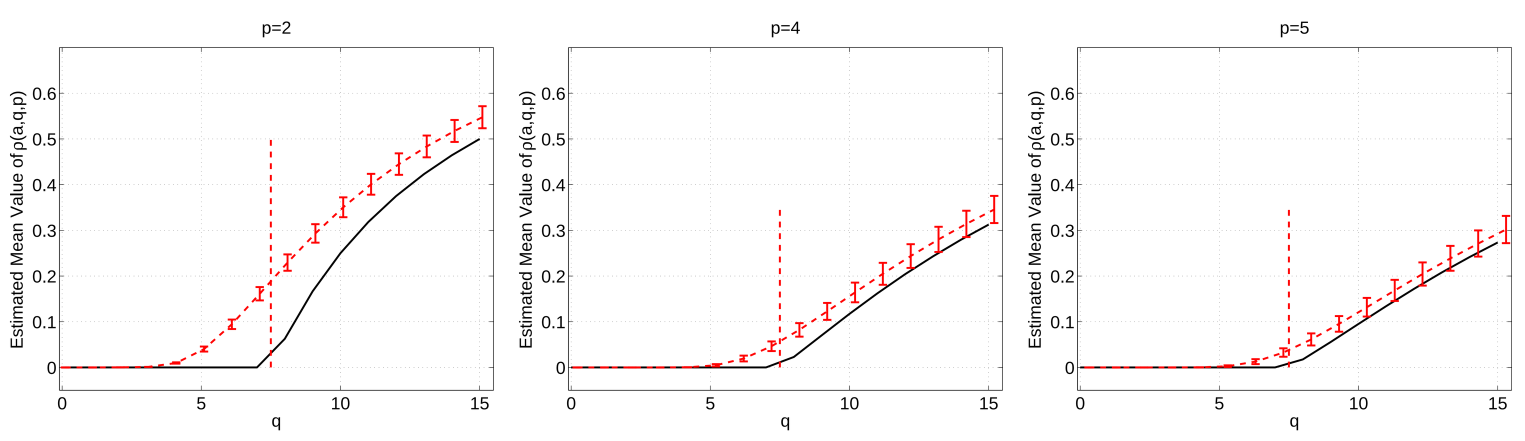
$$X(t) = \lim_{r\to 0} \int_0^t Q_r(s)\, ds.$$

Monte Carlo simulation

Participation ratio
Classical tool for the study of glass transition.

$$\rho(a,q,p) = \frac{\sum_{k=1}^{n_a} |T(a,ka)|^{qp}}{\left(\sum_{k=1}^{n_a} |T(a,ka)|^q\right)^p}.$$

Monte Carlo simulation gives a firm confirmation of the quantitive prediction of q^*.



Solid black line: $\langle \rho(a,q,p) \rangle$; dashed red line: $\langle \rho(a,q,p) \rangle_{MC}$ averaged over Monte-Carlo simulations, with 95% confidence intervals ; red vertical dashed line: position of the critical q^*

Extreme events and correlation [?]

Dominant moment contribution
Saddle point approximation,

$$\langle T(a,t)^q \rangle = \left\langle a^{qh_a(t)} \right\rangle \approx \int_{-\infty}^{+\infty} e^{-|\ln a|[qh+\psi(h)]}\, dh \approx e^{-|\ln a|[qh_m+\psi(h_m)]}.$$

$$q + \psi'(h_m) = 0.$$

Long-range correlation : The number of effectively independent samples n_a scales as

$$n_a \approx \frac{L}{a}.$$

Extreme events cut-off The finite size of the sample leads to a cut-off $h^\dagger$ in the observable negative values of h_a:

$$P(h_a < h_a^\dagger) \approx \frac{1}{n_a},$$

Combining the expression of n_a and p_a leads to :

$$\psi(h^\dagger) = 1$$

Truncated moment A typical value of S_n can be evaluated by a truncated moment

$$S_n(a,q) \approx \int_{h^\dagger}^{+\infty} a^{qh} p_a(h)\, dh.$$

If $h_m(q) < h^\dagger$,

$$S_n(a,q) \approx a^{1+q\lambda'(q^*)} \text{ (Linearization effect)}$$

Critical order : $h_m(q^*) = h^\dagger$

$$1 + q^*\lambda'(q^*) - \lambda(q^*) = 0.$$

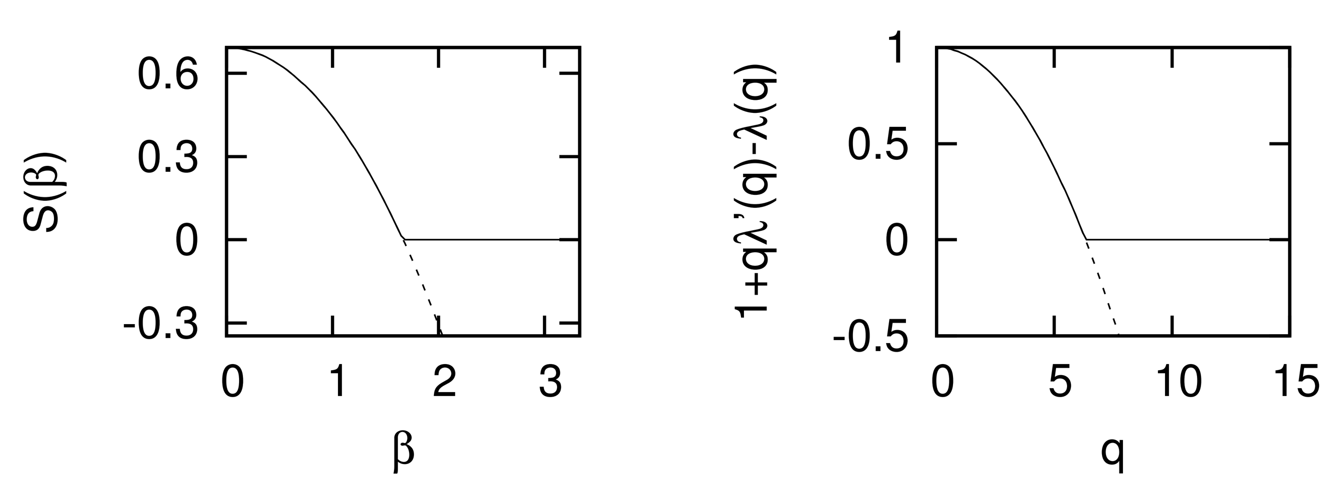
REM Analogy [?]

Formal analogy with the REM model

$$S_n(a,q) = \frac{1}{n}\sum_k T(a,t_k)^q = \frac{1}{n}\sum_k e^{-q|\ln a|h_a(t_k)} \approx Z$$

Z partition function of the REM. $\begin{cases} \text{inverse temperature } \beta \leftrightarrow q \\ \text{energy } E \leftrightarrow -|\ln a| h_a \end{cases}$

REM		Multifractal analysis
Entropy S	$\leftrightarrow$	$1 + q\lambda'(q) - \lambda(q)$
Glass transition at β_c	$\leftrightarrow$	Linearization effect at q^* .



Conclusions

- A fruitful collaboration between statistical physics and multifractal analysis,
- Quantitative prediction of the Linearization effect threshold q^*,
- Monte Carlo simulations confirm the simirlarity between the REM and the multifractal framework,
- q^* is an intrisical characteristic of the process not a experimental limit,
- The Linearization effect is a glassy phase transition!

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