ENS DE LYON

Matrix Products for the Synthesis of Stationary Time Series with a priori Prescribed Joint Distributions

Florian Angeletti, Eric Bertin, Patrice Abry. Physics dept. ENS Lyon, CNRS, Lyon France.
firstname.lastname@ens-lyon.fr

Goal

- Synthesis of time series and random vector
- with a priori prescribed statistical properties
- by product of matrix
- inspired from out of equilibrium statistical Physics
- Hidden Markov Model simulation procedure

Definition and statistical properties

Joint probability density :

$$\mathbb{P}(x_1,\dots,x_N) = \frac{\mathcal{L}\left(\prod_k^N \mathcal{E}\otimes\mathcal{P}(x_k)\right)}{\mathcal{L}\left(\mathcal{E}^N\right)}$$

- $\mathcal{E}\otimes\mathcal{P}(x)_{i,j} = \mathcal{E}_{i,j}\mathcal{P}(x)_{i,j}$ $d\times d$ matrix
- $\mathcal{L}(M) = \mathrm{tr}\left(A^T M\right)$: A positive matrix
- $\mathcal{P}$: probability matrix
- $\mathcal{E}$: positive matrix

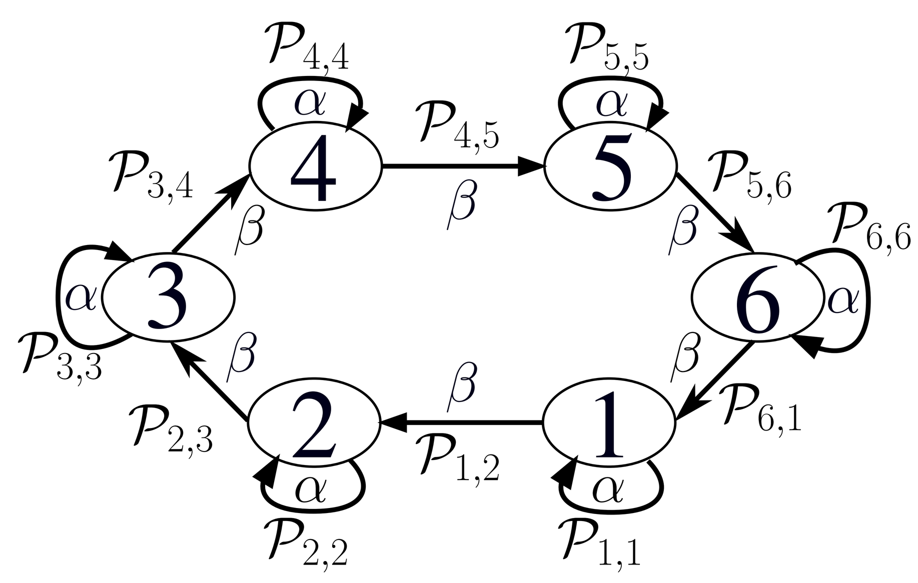
Marginal distribution : $\mathbb{P}(X_k = x) = \frac{\mathcal{L}(\mathcal{E}^{k-1}\mathcal{E}\otimes\mathcal{P}(x)\mathcal{E}^{N-k})}{\mathcal{L}(\mathcal{E}^N)}$

Moments at samples $k_1 < \dots < k_p$ (orders $q_1,\dots,q_p$) :

$$\mathbb{E}\left[\prod_{r=1}^{p} X_{k_r}^{q_r}\right] = \frac{\mathcal{L}\left(\mathcal{E}^{k_1-1}\left(\prod_r^{p-1} M(q_r)\,\mathcal{E}^{k_{r+1}-k_r-1}\right) M(q_p)\,\mathcal{E}^{N-k_p}\right)}{\mathcal{L}(\mathcal{E}^N)}$$

Moment matrices : $M(q) = \mathcal{E}\otimes\int x^q\mathcal{P}(x)dx.$

Synthesis



1. Initialization: Generate the state Γ_0 : $\mathbb{P}(\Gamma_0 = i) = \frac{1}{d}$
2. Iteration on k:
   (a) Choose at random state Γ_k with transition rates $\mathcal{E}_{\Gamma_{k-1},\Gamma_k}$
   (b) Generate X_k according to $\mathcal{P}_{\Gamma_{k-1},\Gamma_k}$

Stationarity

$$\mathcal{A} = \frac{1}{d}\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \beta \\ \beta & 0 & 0 & \alpha \end{pmatrix}, \quad \alpha+\beta = 1$$

Marginal distribution : $\mathbb{P}_k(x) = \mathcal{L}\left(\mathcal{E}\otimes\mathcal{P}(x)\right)$, independent of k.

Moments :

$$\mathbb{E}\left[\prod_{r=1}^{p} X_{k_r}^{q_r}\right] = \mathcal{L}\left(\left(\prod_{r}^{p-1} M(q_r)\,\mathcal{E}^{[k_{r+1}-k_r]-1}\right) M(q_p)\right)$$

Covariance

$$\mathbb{E}\left[X_0^q X_t^q\right] - \mathbb{E}\left[X_0^q\right]\mathbb{E}\left[X_t^q\right] = \sum_{k=1}^{\lfloor d/2\rfloor} m_k \mathfrak{Re}\left\{\mathcal{F}(L_{M(q)})_k \overline{\mathcal{F}(C_{M(q)})_k} e^{-\frac{t-1}{\tau_k}} e^{\imath\frac{2\pi(t-1)}{T_k}}\right\},$$

- $\mathcal{F}$ DFT
- $m_k = 1 + \delta_{2k,d}$
- $(L_{M(q)})_i = \sum_k M(q)_{i,k}$
- $(C_{M(q)})_j = \sum_k M(q)_{k,j}$

Eigenvalues : $\lambda_k = \alpha + \beta\exp(2k\imath\pi/d)$
Correlation lengths : $\tau_k \approx \frac{1}{\alpha\beta(1-\cos\frac{2\imath\pi k}{d})}$
Oscillation periods $T_k = \frac{2\pi}{\arg(\lambda_k)}$

Illustrations

- $d = 6$
- $P(x) = \frac{1}{2}\left(\mathcal{N}_{0,\sigma_1}(x) + \mathcal{N}_{0,\sigma_2}(x)\right)$
- $\tau_1 \approx \frac{3}{2\beta}$
- $\tau_2 \approx \frac{1}{\beta}$
- $\beta = 0.01$
- $M(1) = 0$
- $\tau_3 \approx \frac{1}{2\beta}$

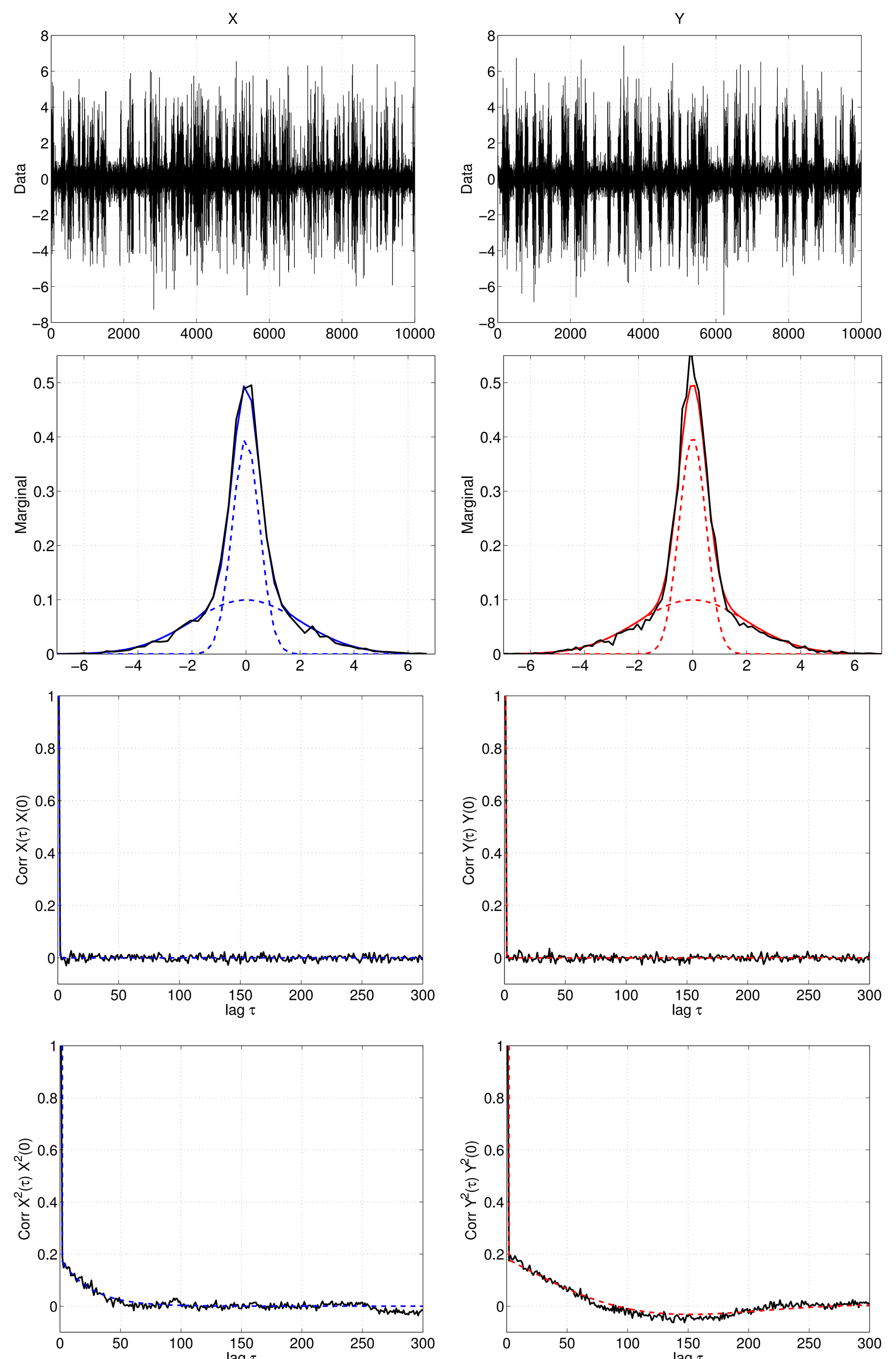
$\underline{X}$, $\tau = \tau_3$ — $\underline{Y}$, $\tau \approx \tau_1$

Realisations :

Probability densities :
Solid lines : marginal,
dashed lines : $\mathcal{P}_{i,j}$

Correlations :

Square correlations :

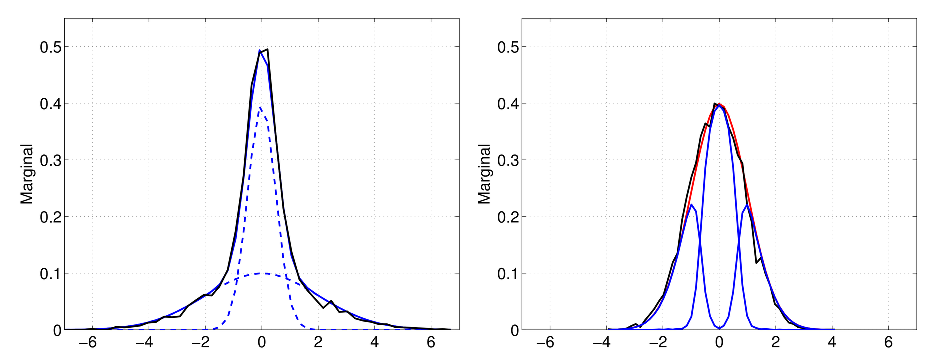


$M(2)$:

$$\begin{pmatrix} \sigma_1 & \sigma_1 & & & & \\ & \sigma_2 & \sigma_2 & & & \\ & & \sigma_1 & \sigma_1 & & \\ & & & \sigma_2 & \sigma_2 & \\ & & & & \sigma_1 & \sigma_1 \\ \sigma_2 & & & & & \sigma_2 \end{pmatrix} \qquad \begin{pmatrix} \sigma_1 & \sigma_1 & & & & \\ & \sigma_1 & \sigma_1 & & & \\ & & \sigma_1 & \sigma_1 & & \\ & & & \sigma_2 & \sigma_2 & \\ & & & & \sigma_2 & \sigma_2 \\ \sigma_2 & & & & & \sigma_2 \end{pmatrix}$$

Automatic design

Prescribed marginal and moments



Random vector

Pair of bivariate random vector $\underline{X}$ and $\underline{Y}$

Same marginal:

- X_1 and Y_1 : gaussian marginal.
- X_2 and Y_2 : gamma marginal.

Same autocovariance : $\mathrm{Cov}\left(\underline{X}\right) = \mathrm{Cov}\left(\underline{Y}\right) = (1-2\alpha)\Delta$

Different joint distributions : $\underline{X} \neq \underline{Y}$.