Sum of matrix-correlated random variables

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Context

- Sequence of random variable $X_1, \dots, X_N$
- Independent? $p(x_1, \dots x_N) = p(x_1) \cdots p(x_N)$?
- Markov chain? $p(x_1, \dots, x_N) = p_1(x_1) p(x_1, x_2) \dots p(x_{N-1}, x_N)$

} short-range correlation.

- How to model long-range correlation?
- Matrix product ansatz : store hidden information
- Influence over the sum $S_N = X_1 + \cdots X_N$?

Definition

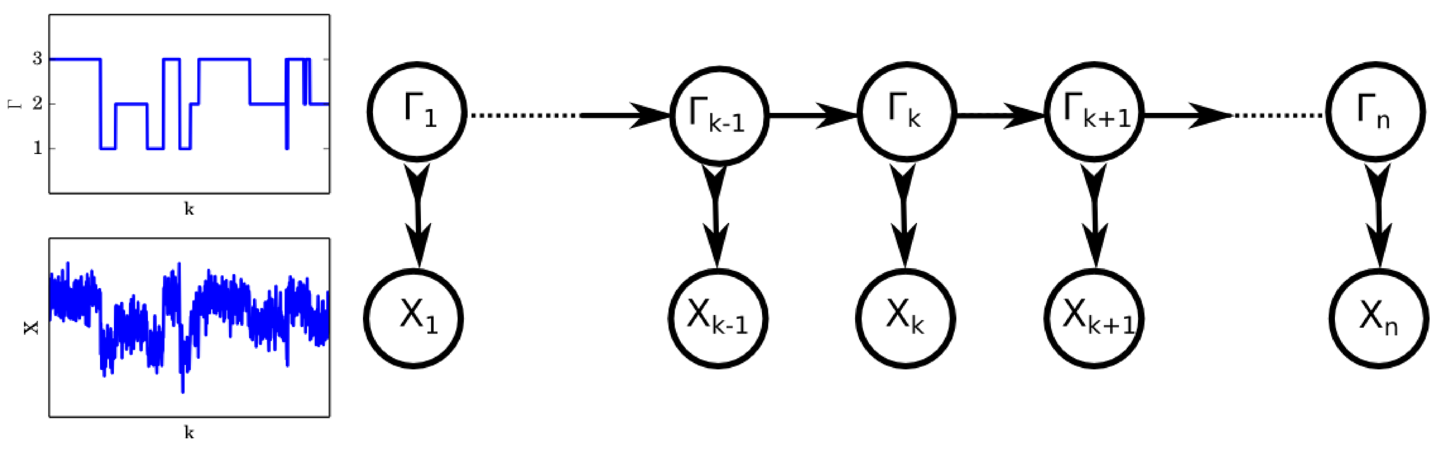
$$P(x_1, \dots, x_N) = \frac{\mathcal{L}\left(\prod_{k=1}^N \mathcal{R}(x_k)\right)}{\mathcal{L}\left(\mathcal{E}^N\right)}, \quad S_N = \sum_{k=1}^N X_k$$

- $\mathcal{R}(x)_{i,j} = \mathcal{E}_{i,j} \mathcal{P}(x)_{i,j}$ $d \times d$ matrix
- $\mathcal{P}$: probability matrix
- $\mathcal{L}(M) = \operatorname{tr}\left(A^T M\right)$: A positive matrix
- $\mathcal{E}$: positive matrix

Hidden Markov Model

- $X \implies (\Gamma, X)$: Separation of randomness into an observable level X and an hidden level Γ
- Γ : Hidden non-homogeneous Markov chain

$$P(\Gamma_1 = i, \Gamma_{N+1} = f) = \mathcal{A}_{if} \frac{(\mathcal{E}^N)_{if}}{\mathcal{L}(\mathcal{E}^N)} \qquad P(\Gamma_{k+1} = j | \Gamma_k = i, \Gamma_{N+1} = f) = \mathcal{E}_{ij} \frac{(\mathcal{E}^{N-k})_{jf}}{(\mathcal{E}^{N-k+1})_{if}}$$

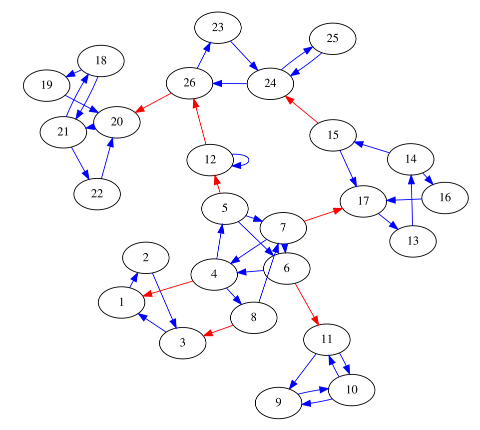


- Observable $\vec{X} = X_1, \dots, X_k$
  - $(X|\Gamma)_k$ are independent: $P(X_k = x | \Gamma) = \mathcal{P}_{\Gamma_k, \Gamma_{k+1}}(x)$

Correlation

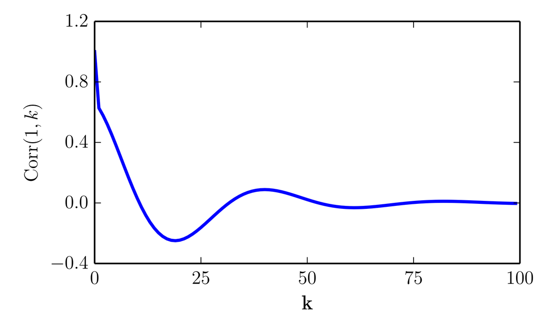
$$k < l, \quad \langle X_k X_l \rangle = \frac{\mathcal{L}\left(\mathcal{E}^{k-1} M(1) \mathcal{E}^{k-l-1} M(1) \mathcal{E}^{N-l}\right)}{\mathcal{L}\left(\mathcal{E}^N\right)}, \quad M(q) = \int x^q \mathcal{R}(x) dx$$

- Graph representation $G(\mathcal{E})$ of $\mathcal{E}$:
  - Edge between i and j in $G(\mathcal{E}) \iff \mathcal{E}_{i,j} > 0$
  - Graphical representation of the Markov chain transitions
  - Two types of transitions : reversible and irreversible
  - Irreversible transitions: ergodicity breakdown



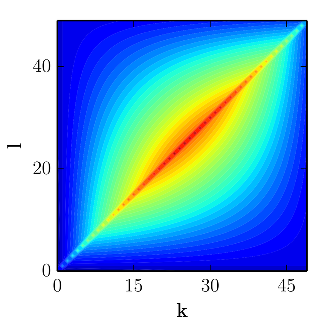
- Reversible transitions:
  - Short-range exponential correlation

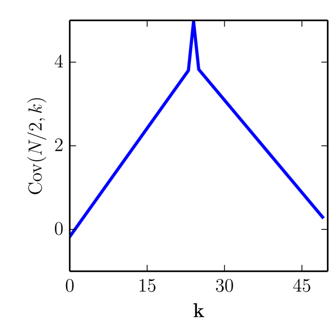
$$\langle X_k X_l \rangle \approx \sum c_e \left(\frac{\lambda_e}{\Lambda}\right)^{|k-l|}$$



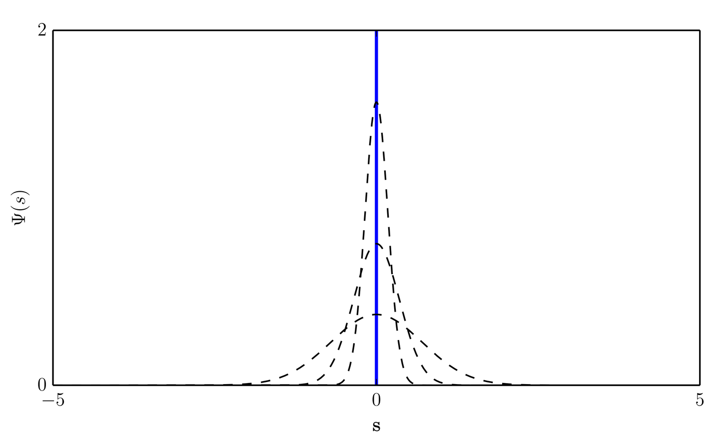
- Irreversible transitions:
  - Long-range piecewise polynomial covariance

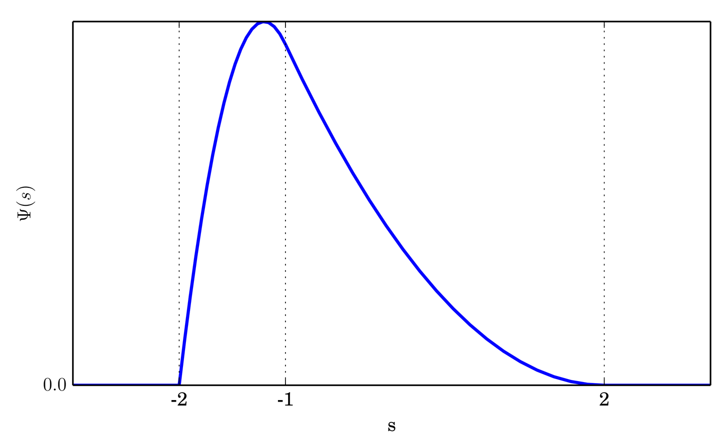
$$\langle X_k X_l \rangle \approx \sum_{r+s+t=\text{const}} c_{r,s,t} \left(\frac{k}{N}\right)^r \left(\frac{k-l}{N}\right)^s \left(1 - \frac{l}{N}\right)^t$$





Generalized law of large numbers





- Law of large numbers for i.i.d. random variables:

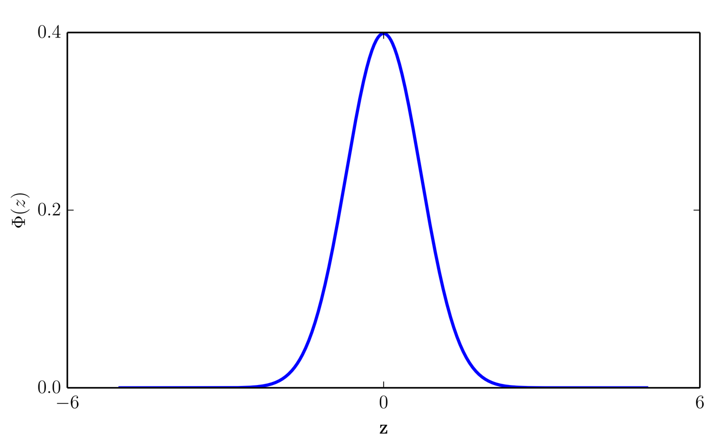
$$\Psi(s) \equiv \lim_{N \to +\infty} p\left(\frac{S_N}{N} = s\right) = \delta(s - \mu), \quad \mu \in \mathbb{R}$$

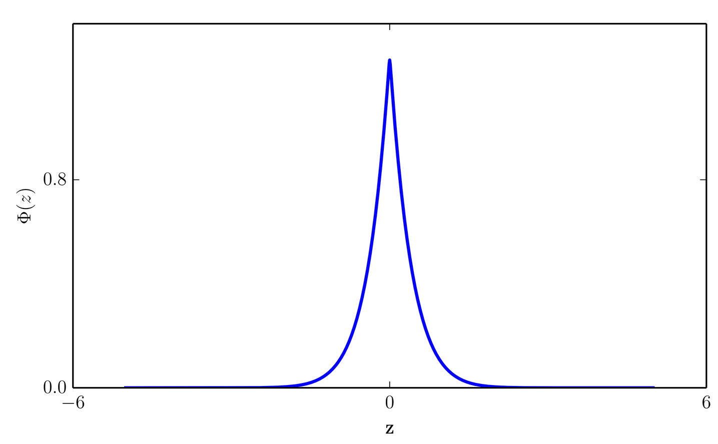
- Generalization of the law of large number : $\Psi(s)$ is no longer a Dirac

$$\Psi(s) = \sum_{\mathcal{C}} p(\mathcal{C}) \int \delta\left(\sum_{k=1}^{l_{\max}} \alpha_k - 1\right) \delta\left(s - \sum_{i,j=1}^{l_{\max}} \alpha_k \mu_{\mathcal{C}_k \mathcal{C}_k}\right) \prod_{k=1}^{l_{\max}} d\alpha_k ,$$

- $\Psi(s)$ is a piecewise polynomial

Generalized central limit theorem





- Central limit theorem for i.i.d. random variables

$$\Phi(z) \equiv \lim_{N \to +\infty} p\left(\frac{S_N - \mu N}{\sqrt{N}} = z\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/\sigma^2}$$

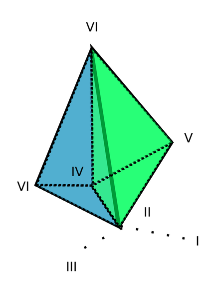
- Generalization of the central limit theorem : $\Phi(z)$ is no longer a gaussian

$$\Phi(z) = \sum_{\mathcal{C}} p(\mathcal{C}) \int \frac{\delta\left(\sum_{k=1}^{l_{\max}} \alpha_k - 1\right)}{\sqrt{2\pi \sum_{k=1}^{l_{\max}} \alpha_k \sigma^2_{\mathcal{C}_k \mathcal{C}_k}}} e^{-z^2/\left[2 \sum_{k=1}^{l_{\max}} \alpha_k \sigma^2_{\mathcal{C}_k \mathcal{C}_k}\right]} \prod_{k=1}^{l_{\max}} d\alpha_k .$$

- $\Phi(z)$ is a discrete mixture of continuous mixture of centered gaussian

Constructive algorithm

- Exact computation of $\Psi(s)$
- Numerical computation of $\Phi(z)$
- Open source code on https://github.com/FAngeletti/Pymatr



Perspectives

- Sum for heavy-tailed distribution: what happens when $\langle X_k^2 \rangle = +\infty$?
- Extreme values statistics: limit distributions for $\max\{X_1, \dots, X_n\}$?
- Application to (TA)SEP: average number of particle?
- Inverse problem: how do we infer $(\mathcal{A}, \mathcal{E}, \mathcal{P})$ from observations?

References

General limit distributions for sums of random variables with a matrix product representation, F. Angeletti, E. Bertin, P. Abry, (2014), arxiv:1406.5016

More on http://perso.quaesituri.org/florian.angeletti